Pre-class Warm-up!!!

Suppose that A is an $m \times n$ matrix (m rows, n columns) and that $Ax = 0$ has a unique solution. Which of the following statements is sometimes false?

a. The columns of A are linearly independent.

b. The columns of A span a space of dimension n.

c. The columns of A are a basis for the space they span.

d. $m \leq n$

Pre-class Warm-up!!!

Which of the following are subspaces of the vector space of all functions R -> R?

a. The set of all functions f such that $f'(1) = 0$.

✓ Yes No

b. The set of all functions f such that $f'(1) = 1$.

Yes No ✓ $(f+g)'(1) = 1+1 = 2 \neq 1$.

c. The set of all functions f such that $f(x) \geq 0$ for all x.

Yes No ✓ If $f(x) > 0$ then $((-1)f)(x) < 0$ so $-f$ is not in the set.

To check if $U \subseteq V$ (a vector space) is a subspace we check

1. If $u, v \in U$ then $u+v \in U$
2. If $u \in U$, $a \in \mathbb{R}$ then $au \in U$.

We check: if f, g are functions with $f'(1) = 0$, $g'(1) = 0$ then $(f+g)'(1) = 0$ and $(af)'(1) = 0$

Note $(f+g)' = f' + g'$.

Section 4.7: General vector spaces

We study:

- vector spaces of matrices like $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
- Vector spaces of functions like $5\sin x - 3\cos x$
- Vector spaces of polynomials like $a_0 + a_1 x + \cdots + a_{17} x^{17}$, $2x^3 - 5x^{10}$
- Solution spaces to homogeneous differential equations

We identify subspaces and find bases in some cases. A more systematic treatment of independence of functions is given in Section 5.1.

You will not be tested on: the justification that the algorithm to find partial fraction decompositions works, in Example 5.

Matrices

3×2 matrices form a vector space

$2\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} - 3\begin{bmatrix}0&-1\\1&1\\2&3\end{bmatrix} = a\ (3\times 2)\text{ matrix}$

Let M_{m,n} denote the set of m x n matrices. This is a vector space.

It has basis the matrices E_{i,j}. $= \begin{bmatrix}0 & 0\\ & 1\\ 0 & 0\end{bmatrix}$

Every matrix is a linear combination of the E_{ij}

The trace of a matrix:

$e.g.\ \begin{bmatrix}1&2\\3&4\end{bmatrix} = 1E_{11} + 2E_{12} + 3E_{21} + 4E_{22}$

If $A = (a_{ij})$ is a square $n\times n$-matrix then $\text{trace } A = a_{11} + a_{22} + \cdots + a_{nn}$ is the sum of entries on the leading diagonal

$\text{trace}\left(2\begin{bmatrix}1&2\\3&4\end{bmatrix} - \begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \text{trace}\begin{bmatrix}1&3\\6&7\end{bmatrix} = 1+7 = 8$

$= 2\,\text{trace}\begin{bmatrix}1&2\\3&4\end{bmatrix} - \text{trace}\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$= 2\cdot 5 - 2 = 8$

Question like Section 4.7, 1-4.
Which of the following are subspaces of M_{3,3}?

$\text{trace}(A+B) = \text{trace } A + \text{trace } B = 0+0=0$

a. Matrices of trace 0. Yes

b. Matrices of trace 5. Yes No ✓

c. Matrices of determinant 1.

d. Upper triangular matrices. Yes ✓ No

Not a subspace: $\det\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right) = \det\begin{bmatrix}0&0\\0&0\end{bmatrix} = 0$

$\begin{bmatrix}*&*&*\\0&*&*\\0&0&*\end{bmatrix} + \begin{bmatrix}\square&\square&\square\\0&\square&\square\\0&0&\square\end{bmatrix} = \begin{bmatrix}\cdot&\cdot&\cdot\\0&\cdot&\cdot\\0&0&\cdot\end{bmatrix}$

$a\begin{bmatrix}\cdot&\cdot&\cdot\\0&\cdot&\cdot\\0&0&\cdot\end{bmatrix} = \begin{bmatrix}\cdot&\cdot&\cdot\\0&\cdot&\cdot\\0&0&\cdot\end{bmatrix}$

Polynomials

Example (like example 4 from Section 4.4 and Example 6 in Section 4.7):

Let V $= P_3$ be the set of polynomials

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$

(a) Show that V has dimension 4.
(b) Show that 1, 1+x, x+x^2, x^2+x^3 is a basis for V.

a. We show: $1, x, x^2, x^3$ is a basis for V.

They span: $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$= a_0 1 + a_1 x + a_2 x^2 + a_3 x^3$

They are independent:

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$

$\iff a_0 = a_1 = a_2 = a_3$

because two polynomials are equal $\iff$ all their coefficients are the same.

Polynomials

In the book, P_n denotes the set of all polynomials of degree $\leq n$. It is a space of dimension $n + 1$.

Questions like Section 4.7, 9-12.

Which of the following are subspaces of P_5 ?

a. The polynomials $p(x)$ with $a_2 = 0$.

Yes No

b. The polynomials $p(x)$ with $a_2 = 1$.

Yes No

Like Section 4.7 questions 13-16 as well as questions in 5.1. In 5.1 we learn a different approach to testing if functions are independent.

Which of the following sets of functions are independent?

a. e^x and sin x

b. ln x and ln(x^2)

c. cos x + 2 sin x and 2 cos x + sin x.

d. e^x, sin x and 1.

Two vectors are dependent $\Leftrightarrow$ one is a scalar multiple of the other.

a. Is e^x a scalar multiple of $\sin x$?

No. They are independent

b. Note $\ln(x^2) = 2\ln x$

so $\ln(x^2)$, $\ln x$ are dependent.

c. They are independent.

d. We show ~~they are~~ independent.

If $ae^x + b\sin x + c = 0 \ \forall x$

then for values x_1, x_2, x_3 of x the vectors $\begin{bmatrix} e^{x_1} \\ e^{x_2} \\ e^{x_3} \end{bmatrix}$ $\begin{bmatrix} \sin x_1 \\ \sin x_2 \\ \sin x_3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are dependent.

Take $x_1 = 0$, $x_2 = \pi/2$, $x_3 = \pi$.

$\begin{bmatrix} 1 \\ e^{\pi/2} \\ e^{\pi} \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are independent.

Therefore the functions are independent.

Like Section 4.7 question 25:

Find a basis for the solution space of $y'' + 3y' = 0$.

Question:
Do the following matrices form a basis for M_{2,2} ?

a. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Yes

No

b. What about $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?

Yes

No